Performance Improvement of Algebraic Multigrid Solver by Vector Sequence Extrapolation

Aleksandar Jemcov[1], Joseph P. Maruszewski[1], and Hrvoje Jasak[2]

[1] *Development Department, Ansys/Fluent Inc., Lebanon, NH, 03766, USA*
[2] *Wikki Ltd., 33 Palmerston House, 60 Kensington Place, London, W8 7PU, UK and*
Faculty of Mechanical Engineering and Naval Architecture, University of Zagreb, Croatia

Email: *aj@fluent.com*

ABSTRACT

Algebraic Multigrid Method (AMG) performance improvement by vector sequence extrapolation is examined. Projective Forward Extrapolation (PFE), Minimal Polynomial Extrapolation (MPE) and Reduced Rank Extrapolation (RRE) are applied to the AMG resulting in a hybrid approach, vector extrapolated AMG. The impact of vector sequence extrapolation is shown to improve performance of the AMG in number of cycles and execution time, resulting in three new methods: PFE-AMG, MPE-AMG and RRE-AMG. Computational results of the application of vector extrapolated AMG to sparse matrices arising from discretization of fluid flow equations are presented showing performance improvements compared to the traditional AMG.

1 INTRODUCTION

The role of the algebraic multigrid method as an accelerator of stationary iterative methods [1, 2, 3] in implicit finite volume and finite element methods is well established. Stationary iterative methods are known to have poor convergence performance as the computational grid increases in size. This is mainly due to the unfavorable eigenvalue spectrum of the iteration matrix of the stationary iterative methods produced by the increase in the grid size [1, 4]. The AMG solver relies on the smoothing properties of the underlying stationary method to remove high frequency errors by projecting the residual and the linear system matrix to a sequence of progressively coarser AMG levels [1, 2]. Coarse matrices are created algebraically by constructing the restriction and prolongation operators in the process of agglomeration. By solving a sequence of problems on a hierarchy of coarse matrices, error correction is computed and transferred back to the fine level where the solution is improved. Acceleration of the solution is obtained by using the stationary iterative methods on each coarse level to remove high frequency error visible on this level, thus creating the efficient acceleration strategy.

As the problem size increases, AMG can have difficulties in reducing the linear system residuals to a specified tolerance due to an increased condition number of the system. Moreover, the AMG method may even diverge if some of the iteration matrix eigenvalues lie outside of the unit circle. In this case, high frequency errors on coarse AMG levels do not get eliminated, instead they get amplified. One remedy to this problem is in the use of special projection techniques such as the Recursive Projection Method (RPM(n)) [5, 6]. In this method, unstable or slowly converging subspace is identified and stable and unstable projectors are constructed so that the solution vector and the fixed-point function of the iterative method can be projected onto stable and unstable subspaces. Special methods to solve the unstable part of the algorithm are used while AMG is applied on the stable part. The RPM(n) method stabilizes the AMG method and improves its performance [6]. An alternative approach uses Krylov space methods [7, 4] for the solution of the linear system of equations. Many Krylov space solvers are available [7, 4] and their fundamental characteristic is the non-stationary nature of the iterative process. Depending on the problem, the Krylov space may have a large dimension requiring additional computer memory. The methods are usually used in conjunction with a preconditioning technique [7, 8]. The role of a preconditioner is to create a more favorable iteration matrix spectrum which effectively reduces the Krylov space dimension, accelerating convergence. In general, the performance of the Krylov space solvers in FVM and FEM analysis depend more on the quality of a preconditioner than the choice of a iterative solver. An optimal preconditioner choice is dependent upon the type of the problem, many preconditioners are described in literature [7]. It is interesting to note that the AMG method can be used as an efficient preconditioner for Krylov space methods [8] thus providing an efficient preconditioned solver with excellent convergence properties.

An alternative approach to improving AMG performance is found in the use of vector extrapolation methods. Vector extrapolation methods [9] are based on the idea of solution extrapolation. Depending on the way in which the extrapolation is performed, different methods result. One simple approach is to use Projective Forward Extrapolation (PFE) [10]. It utilizes a combination of efficient smoothers and first order extrapolation to predict the new solution. In this case, coefficients of the extrapolation polynomial as well as the polynomial itself are very simple. Another vector extrapolation approach is in the use of Minimal Polynomial Extrapolation and Reduced Rank Extrapolation. Both utilize a Krylov space of a small dimension and approximate the solution in it. This approach results in a non-linear polynomial representation that is closely related to Krylov space methods.

Here, we will investigate the use of vector extrapolation methods for acceleration of AMG when the matrix of the linear system is of large dimension. Convergence properties of vector extrapolated AMG is examined when applied to the solution of the pressure equation obtained in Finite Volume Discretization of the incompressible Navier-Stokes equations.

2 FIXED POINT ITERATIONS FOR LINEAR SYSTEMS OF EQUATIONS

Consider a system of linear equations in matrix form

$$\mathbf{A}\mathbf{x} = \mathbf{b}. \tag{1}$$

Here $\mathbf{A}$ is a matrix of dimensions $N \times N$, $\mathbf{x}$ is a vector of dimension N and $\mathbf{b}$ is a right-hand-side vector of the same dimension. We are interested in finding the solution $\mathbf{x}$ that satisfies Eq. (1).

Stationary iterative methods proceed by devising a decomposition of the matrix $\mathbf{A}$ into two matrices $\mathbf{M}$ and $\mathbf{N}$ [7, 1]

$$\mathbf{A} = \mathbf{M} - \mathbf{N}. \tag{2}$$

Matrix $\mathbf{M}$ is usually called preconditioning matrix whereas $\mathbf{N}$ is called defect matrix. Different choices of the split, Eq. (2), give different stationary methods such as Gauss-Seidel, Jacobi, SOR, SSOR, etc. Once the split is known, fixed-point algorithm for any stationary method is given by

$$\mathbf{x}^{(\nu+1)} = \mathbf{R}\mathbf{x}^{(\nu)} + \mathbf{M}^{-1}\mathbf{b}. \tag{3}$$

Here ν is iteration index and $\mathbf{R}$ is the iteration matrix defined as

$$\mathbf{R} = \mathbf{M}^{-1}\mathbf{N}. \tag{4}$$

An important property of the iteration matrix is that it governs error propagation in the fixed point iterations, Eq. (3). This can be seen if error $\mathbf{e}^{\nu}$ is defined as the difference between the current solution iterate $\mathbf{x}^{(\nu)}$ and the fixed point $\mathbf{x}^{*}$:

$$\mathbf{e}^{(\nu)} = \mathbf{x}^{(\nu)} - \mathbf{x}^{*}. \tag{5}$$

This definition is substituted in Eq. (3) to obtain the error propagation equation:

$$\mathbf{e}^{(\nu+1)} = \mathbf{R}\mathbf{e}^{(\nu)}. \tag{6}$$

The relation in Eq. (6) can be extended to link the initial $\mathbf{e}^{(0)}$ and current $\mathbf{e}^{(\nu)}$ errors through the following recursion:

$$\mathbf{e}^{(\nu+1)} = \mathbf{R}^{\nu}\mathbf{e}^{(0)}. \tag{7}$$

The importance of the recursion relation, Eq. (6) is in the fact that the fixed point method in Eq. (3) generates a Krylov subspace

$$\mathcal{K} = span(\triangle\mathbf{x}^{(0)}, \triangle\mathbf{x}^{(1)}, \cdots, \triangle\mathbf{x}^{(n)}). \tag{8}$$

where $\triangle\mathbf{x}^{(\nu)}$ is defined as

$$\triangle\mathbf{x}^{(\nu)} = \mathbf{x}^{(\nu)} - \mathbf{x}^{(\nu-1)}. \tag{9}$$

Sometimes the Krylov space, Eq. (8) is called the correction space. Given the correction space, fixed point $\mathbf{x}^{*}$ can be approximated from the current solution $\mathbf{x}^{(\nu)}$ using the following relation

$$\mathbf{x}^{*} = \mathbf{x}^{(\nu)} + \sum_{\nu} \alpha_{\nu} \triangle\mathbf{x}^{(\nu)}. \tag{10}$$

The relation in Eq.(10) is used here to accelerate the AMG method. Depending on how the coefficients α_i are determined, different methods result.

3 ALGEBRAIC MULTIGRID METHOD

To create a sequence of corrections $\triangle\mathbf{x}^{(\nu)}$ that form the correction space $\mathcal{K}$, some fixed-point process must exist. Since we are interested in accelerating iterative solvers for systems of linear equations, AMG is used to create the correction space. The AMG method is an acceleration strategy for basic stationary methods such as Gauss-Seidel and Jacobi in which a hierarchy of progressively coarser matrices are constructed and coarse correction problems are solved [1, 2]:

$$\mathbf{A}^n\mathbf{e}^n = \mathbf{r}^n. \tag{11}$$

Here, $\mathbf{A}^n$ is a matrix, $\mathbf{e}^n$ is the correction vector, and $\mathbf{r}^n$ is residual at multigrid level n. Once the correction vector is known, next level residual and correction can be updated through projection from coarse to fine level until the finest level is reached where the current iterate $\mathbf{x}^{(\nu)}$ can be corrected

$$\mathbf{x}^{(\nu)} \leftarrow \mathbf{x}^{(\nu)} + \mathbf{P}_{n+1}^{n}\mathbf{e}^{n+1} \tag{12}$$

Here $\mathbf{P}_{n+1}^{n}$ is the prolongation operator constructed together with the restriction operator $\mathbf{R}_{n}^{n+1}$ during the creation of a coarse matrix hierarchy through agglomeration process. The agglomeration process computes restriction operator by creating the coarse stencils of rows of matrix $\mathbf{A}$ so that a low frequency error on fine mesh becomes a high frequency error on coarse mesh [3, 1, 2].

Acceleration properties of AMG rely on the effective use of the underlying smoother [1, 2]. At each AMG level, several smoothing steps are taken during the cycle and the ability of the smoother to remove the high frequency error from the current iterate $\mathbf{x}^{(\nu)}$ is critical to success of the AMG iteration. However, if the underlying matrix $\mathbf{A}$ violates the M-matrix property [4], the smoothing operation will diverge, causing the AMG cycle to diverge as well. Furthermore, even if matrix $\mathbf{A}$ does not violate the M-matrix property, the AMG iterations may fail to provide high rate of convergence. This happens when the iteration matrix, Eq. (4), has an unfavorable eigenvalue spectrum with many eigenvalues close to the edge of the unit circle in the complex plane [6]. An unfavorable spectrum of the iteration matrix $\mathbf{R}$ is usually caused by physical models, the discretization scheme used to obtain matrix coefficients, a low quality computational grid, etc. In such situations, AMG exhibits stalled or slow convergence, or divergence.

One solution to this problem is to modify AMG by implementing the recursive projection method (RPM(n)) [6, 5]. Here we shall investigate extrapolation techniques for performance improvements of AMG.

4 PROJECTIVE FORWARD EXTRAPOLATION

As shown above, slow convergence is caused by the outlying eigenvalues of the iteration matrix $\mathbf{R}$. If the spectrum of the iteration matrix can be divided into clusters of eigenvalues based on the error frequency, it is plausible to expect that the fixed-point iterations will quickly remove high frequency errors and spend more time iterating on the smooth part of the error. Assuming the overall process converges, once the high frequency error is removed, iterations will be approaching the fixed-point at slower rates.

Projective forward extrapolation (PFE) is based on a simple linear extrapolation of the current iterate when high frequency error is removed [10]. The idea of the PFE method is that a simple linear extrapolation can be used to approximate the fixed-point

$$\mathbf{x}^* = \mathbf{x}^{(\nu)} + \alpha \triangle \mathbf{x}^{(\nu)}. \tag{13}$$

Coefficient α is usually chosen to be equal to 1, although other values can be used [10]. The correction space dimension is equal to 2, making this algorithm attractive in practical computations. Before extrapolation in Eq. (13) can be done, smoothing steps must be performed to remove fast pseudo-transients associated with high frequency error. If this error is not removed, the PFE algorithm will amplify them causing convergence problems.

Since the dimension of the correction space is small, PFE-AMG algorithm is implemented with restarts. First n AMG cycles are performed to remove fast pseudo-transients. After that, an extrapolation step is performed followed by n AMG cycles. This approach allows removal of high frequency error associated with the extrapolation step while still benefiting from the approximation of low frequency through the process of extrapolation.

5 MINIMAL POLYNOMIAL EXTRAPOLATION

The PFE-AMG algorithm assumed a particularly simple form of extrapolation function. In addition, the extrapolating coefficient was chosen to be equal to 1, effectively restricting the accuracy of extrapolation. It is natural to ask if other choices of interpolating coefficients and a higher dimension of correction space $\mathcal{K}$ can be used?

The answer is obtained by observing that the fixed point $\mathbf{x}^*$ at convergence can be expressed using Eq. (10) and Eq. (6):

$$\mathbf{x}^* = \mathbf{x}^* + \sum_{\nu} \alpha_\nu \mathbf{e}^\nu. \tag{14}$$

Strictly speaking, the relation in Eq. (14) is valid only at a fixed point in infinite precision arithmetic. In other words, extrapolation coefficients must be chosen so that once the fixed point is reached, they remove the error from the solution:

$$\sum_\nu \alpha_\nu \mathbf{e}^\nu = 0. \tag{15}$$

If we introduce the minimal polynomial $P_M(\mathbf{R})$ of the iteration matrix, then the following identity is obtained:

$$P_M(\mathbf{R})\triangle\mathbf{x}^0 = \sum_\nu^M c_\nu \mathbf{R}^\nu \triangle\mathbf{x}^{(0)} = 0. \tag{16}$$

Here $P_M(\mathbf{R})$ is a monic polynomial with coefficients c_ν that satisfy

$$\sum_\nu^M c_\nu = 1. \tag{17}$$

Further manipulation of expression in Eq. (16) leads to

$$\sum_\nu^M c_\nu \mathbf{R}^\nu \triangle\mathbf{x}^{(0)} = (\mathbf{I}-\mathbf{R})\sum_\nu^M c_\nu \mathbf{e}^{(\nu)} = 0 \tag{18}$$

and, since $\mathbf{I}-\mathbf{R}$ has a full rank, we obtain

$$\sum_\nu^M c_\nu \mathbf{e}^{(\nu)} = 0. \tag{19}$$

Coefficients c_ν are obtained by requiring that the L_2 norm of the approximation in Eq. (14) is minimized

$$\mathit{argmin}_{\alpha_\nu} \left\| \mathbf{x}^{(\nu)} + \sum_\nu \alpha_\nu \triangle\mathbf{x}^{(\nu)} \right\|_2 . \tag{20}$$

In practical terms, coefficients c_ν are obtained by solving the over-determined linear problem

$$\mathbf{U}_{M-1}\tilde{c} = \triangle\mathbf{x}^{(M)}. \tag{21}$$

with the constraint

$$c_M = 1 \tag{22}$$

Here $\mathbf{U}_{M-1}$ is a rectangular matrix of dimension $(M-1)\times N$

$$\mathbf{U}_{M-1} = \left[\triangle\mathbf{x}^{(0)}, \triangle\mathbf{x}^{(1)}, \cdots, \triangle\mathbf{x}^{(M-1)}\right] \tag{23}$$

and $\tilde{c}$ is a vector of coefficients of size $M-1$. Once coefficients c_ν are obtained, the constraint in Eq. (22) is satisfied and coefficients c_ν are scaled as follows to obtain coefficients α_ν:

$$\alpha_\nu = \frac{c_\nu}{\sum_\nu^{M-1} c_\nu}. \tag{24}$$

With coefficients α_ν, extrapolation according to Eq. (14) can be performed.

The MPE method applied to AMG is implemented in a similar fashion as PFE-AMG. However, here the dimension of the correction space can be chosen to be larger, producing better extrapolation. The cost of solving the problem involves computation and storing of differences, Eq. (9), and solution of the over-determined linear problem, Eq. (21). Since the size of the over-determined problem is $M \times M$, it does not add to the cost of the algorithm. The biggest cost is in storing the difference vectors as it adds to the total memory used by the solver. Therefore, the cost of the MPE-AMG will be acceptable if the size of the correction space is not too large. This problem is overcome by using the restart technique. A correction space is selected of some fixed dimension and, as soon as the extrapolation is performed, iterations are restarted and a new correction space is formed.

It is also interesting to note that mathematically the MPE method is equivalent to the Full Orthogonal Method (FOM) [7, 9] and with the combination with AMG, MPE-AMG is mathematically equivalent to AMG preconditioned FOM. This equivalence applies only in the case of MPE being applied to a linear systems of equations.

6 REDUCED RANK EXTRAPOLATION

Reduced Rank Extrapolation (RRE) is similar to MPE but uses a different constraint for the minimization problem, Eq. (21). Here the constraint is

$$\sum_\nu^M \alpha_\nu = 1 \tag{25}$$

and the minimization problem becomes

$$\mathbf{U}_M\tilde{\alpha} = 0 \tag{26}$$

subject to the constraint in Eq. (25). Here $\tilde{\alpha}$ is a vector of extrapolating coefficients.

Once the solution to the problem in Eq. (26) is obtained, it is possible to extrapolate the solution according to expression in Eq. (14). Similarly to MPE-AMG, RRE-AMG is implemented with restarts in order to keep the size of the correction space reasonable. The choice of the constraint, Eq. (25) corresponds to the GMRES method. In combination with AMG, RRE-AMG is effectively a preconditioned GMRES. Since GMRES is guaranteed to converge if the basis of Krylov space is large enough [7], the restarted RRE-AMG will also share similar robust properties. This makes RRE-AMG suitable for problems that involve matrices with non-M-matrix property where AMG diverges.

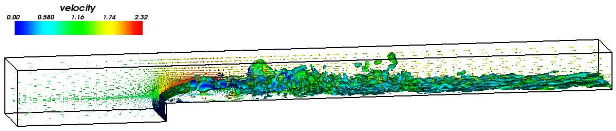

Figure 1: Enstrophy colored by turbulent kinetic energy with velocity vector field in the central plane

7 RESULTS

Here we investigate two linear system problems obtained by the finite volume discretization of the pressure equation on an unstructured tetrahedral mesh. New solvers, FPE-AMG, MPE-AMG, and RRE-AMG are compared to a classical AMG solver in terms of number of iterations and CPU time needed to reduce initial residual by eight orders of magnitude. Two different problems were considered in order to establish the performance of proposed method with respect to different flow physics. In both cases a cell centered second order in space and time finite volume scheme is used for the discretization of the governing equations.

The first test case is the large eddy simulation of the flow over a forward facing step with Reynolds number $Re = 10^5$. The flow field is rich with flow structures and a representative snapshot of instantaneous flow field is shown on Fig. (1). The difficulties associated with this problem result from an aggressive grading of the mesh towards the wall in order to resolve near-wall turbulence interactions. The ratio of the largest to smallest cell volume is over 5000 resulting in increased stiffness of the problem. Several AMG control parameters were varied to examine their influence on convergence properties including the type of the AMG cycle, group size, and smoother. Furthermore, PFE, MPE and RRE extensions of AMG algorithm used the same parameters to form a complete picture of the convergence properties of the examined algorithms.

Convergence histories for different selections of multigrid control parameters and different solvers for the large eddy simulation of the forward facing step are shown in Fig. (2) and Fig. (3). Fig. (2) shows convergence history for four methods using the AMG V-Cycle strategy using a Gauss-Seidel smoother with zero pre-sweeps and two post-sweeps. In all cases agglomeration was performed by limiting the size of coarse groups to two resulting in a gradual coarsening with the ratio of the number of cells on consecutive multigrid levels equal to powers of two. It can be seen that AMG requires 149 V-Cycles to reduce the norm of the residual by eight orders of magnitude. PFE-AMG on the other hand, needs only 133 cycles, whereas MPE-AMG needs 107 and RRE-AMG needs 105 cycles to achieve the same level of residual reduction. In both cases when the MPE and RRE methods of extrapolation were used, the size of the correction space was 5. Choosing a larger dimension of the correction space results in a lower number of cycles needed to reduce the residual to the same level in exchange for longer computation time. All three extrapolation algorithms produce beneficial effects by reducing the overall number of iterations. The most successful one was RRE-AMG which required 105 cycles to converge to the given tolerance. It is also important to consider the computation time, in addition to the total number of iterations, required to converge the linear problem. Results of computations using the selection of the AMG parameters according to Fig. (2) are summarized in Table (1). The largest reduction in the number of the AMG cycles required to achieve the desired level of convergence is brought by RRE-AMG since it required 45 fewer cycles than the original AMG method. At the same time, the performance

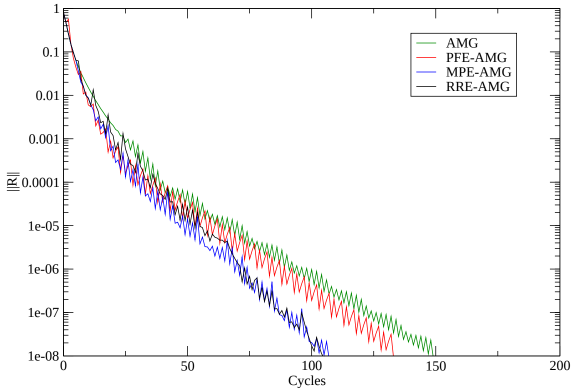

Figure 2: Multigrid parameters for LES problem: V-Cycle, Group Size 2, Gauss-Seidel smoother, 0 pre-sweeps and 2 post-sweeps.

Method	AMG Cycles	Time [s]
AMG	149	72.53
PFE-AMG	133	65.62
MPE-AMG	107	57.35
RRE-AMG	105	56.07

Table 1: Summary of results for Fig. (2).

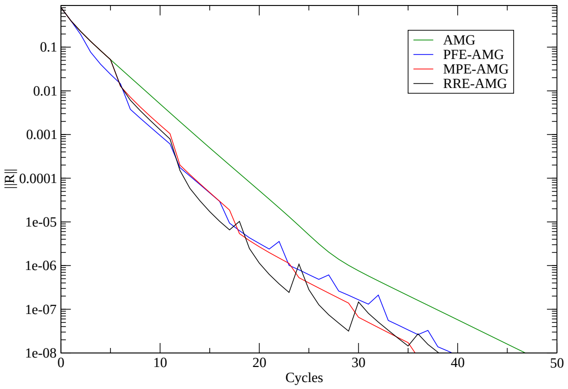

Figure 3: Multigrid parameters for LES problem: W-Cycle, Group Size 4, ILU(0) smoother, 0 pre-sweeps and 2 post-sweeps.

Method	AMG Cycles	Time [s]
AMG	47	27.95
PFE-AMG	40	24.20
MPE-AMG	36	22.90
RRE-AMG	39	24.58

Table 2: Summary of results for Fig. (3).

improvement measured in computation time was approximately 22%. This improvement was achieved at the expense of increased memory usage which in this case required storage of 8 additional solution vectors. In contrast to RRE-AMG, PFE-AMG achieved modest performance improvement by reducing the overall number of AMG cycles by 16 with a performance increase of approximately 9%. However, this performance improvement was achieved by only storing one additional solution vector.

The influence of increased group size from 2 to 4, changing the cycle type from V-Cycle to W-Cycle, and changing the type of the smoother from Gauss-Seidel to ILU(0) is shown in Fig. (3). Changing the smoother from Gauss-Seidel to ILU(0) brings a substantial decrease in computation time and number of the AMG cycles. Extrapolation does have an effect of reducing the number of the AMG cycles required, although this performance improvement is of the order of 10%.

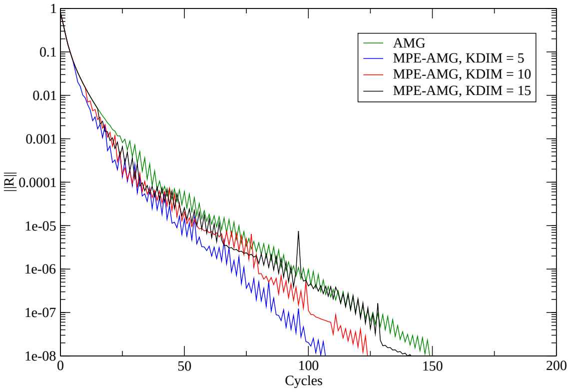

Figure 4: MPE-AMG parameters for LES problem: V-Cycle, Group Size 2, Gauss-Seidel smoother, 0 pre-sweeps and 2 post-sweeps.

Dimension	AMG Cycles	Time [s]
0	149	72.53
5	107	57.35
10	124	68.88
15	142	79.87

Table 3: Summary of results for Fig. (4).

The influence of the correction space dimension on convergence properties of the MPE-AMG algorithm is shown in Fig. (4) and Fig. (5). Corresponding data is shown in Table (3) and Table (4). Results show that increasing the dimension of the correction space for the MPE-AMG algorithm does not necessarily lead to an improvement of convergence speed or to a decrease of computational time. This observation is in agreement with the behavior of Krylov space solvers, where it is important to capture the dimension of the Krylov space correctly [7].

A similar study is performed for the RRE-AMG algorithm where the dimension of the correction space was varied from 0 to 15 in increments of 5.

Dimension	AMG Cycles	Time [s]
0	47	27.95
5	36	22.9
10	45	29.24
15	46	29.56

Table 4: Summary of results for Fig. (5).

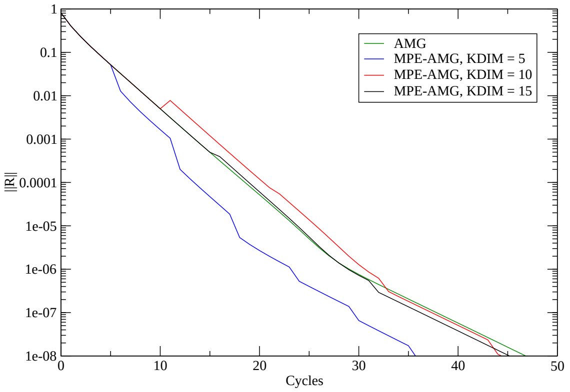

Figure 5: MPE-AMG parameters for LES problem: W-Cycle, Group Size 4, ILU(0) smoother, 0 pre-sweeps and 2 post-sweeps.

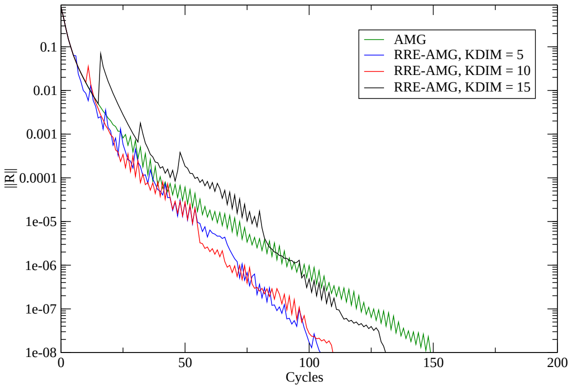

Figure 6: RRE-AMG parameters for LES problem: V-Cycle, Group Size 2, Gauss-Seidel smoother, 0 pre-sweeps and 2 post-sweeps.

Dimension	AMG Cycles	Time [s]
0	149	72.53
5	105	56.07
10	110	62.02
15	131	76.86

Table 5: Summary of results for Fig. (6).

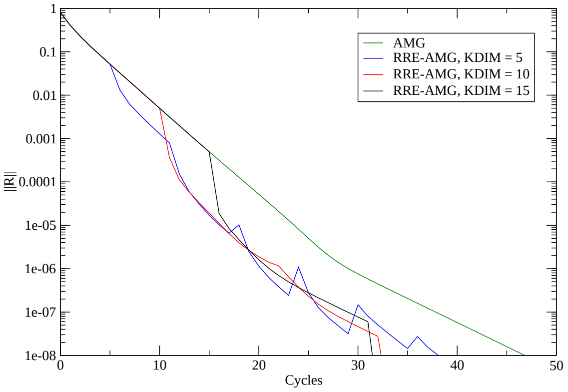

Figure 7: RRE-AMG parameters for LES problem: W-Cycle, Group Size 4, ILU(0) smoother, 0 pre-sweeps and 2 post-sweeps.

Dimension	AMG Cycles	Time [s]
0	47	27.95
5	39	24.58
10	33	21.86
15	32	22.27

Table 6: Summary of results for Fig. (7).

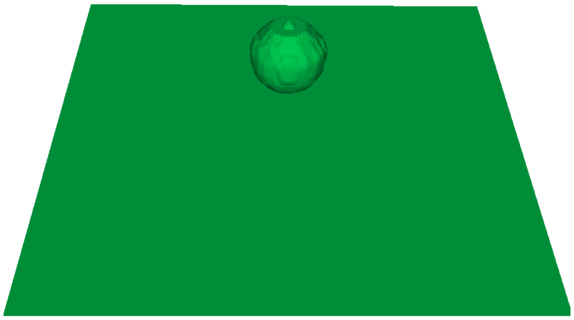

Figure 8: Droplet free surface at time $t = 2.77325E-05\,s$

As with the MPE-AMG algorithm, increasing the size of the correction space does not necessarily reduce the computation time nor the number of the AMG cycles required to achieve a desired reduction in the norm of the residual. Data in Table (5) and Table (6) clearly show that increasing the dimension of the correction space is beneficial when ILU(0), W-Cycle and group size of 4 are used in computations.

A second numerical experiment involves a liquid droplet splash onto a flat surface modeled by an adaptive time-stepping finite volume scheme with a surface tracking algorithm. The solution algorithm tracks the position of the free surface using a volume fraction variable, with a special discretization which preserves a sharp interface [11, 12]. The test case represents impact of a small droplet into wall film with dominant surface tension effects and a density ratio of 1000 between the two phases. The algorithm is a segregated pressure based solver and we are looking at the pressure equation. The mesh has 1296000 cells in 3-D with moderate refinement towards the wall. The difficulties in the pressure solution are caused by the density ratio of 1000 and surface tension forces. Figures (8) and (12) show the free surface shortly before and at the time of impact of the droplet at a flat wall.

Similar to the large eddy simulation over the forward facing step, a parametric analysis of different controlling parameters for four algorithms is performed with the goal of establishing the behavior of the algorithms with respect to those parameters. Figures (13) and (14) together with tables (7) and (8) show the influence of the multigrid cycle type, group size, and type of the smoother on the convergence properties of the four

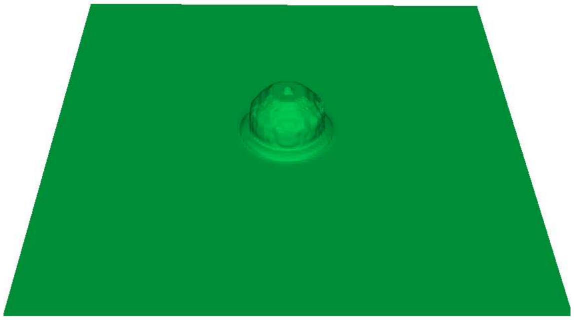

Figure 9: Droplet free surface at time $t = 4.01785E-05\,s$

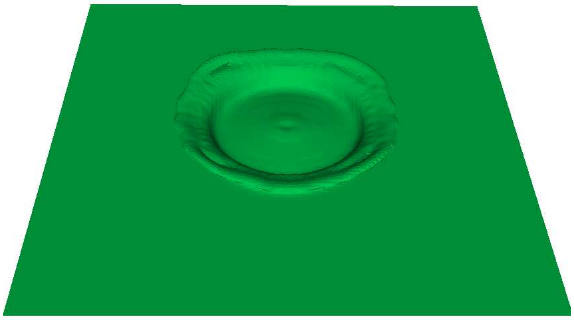

Figure 10: Droplet free surface at time $t = 6.80427E-05$

Figure 11: Droplet free surface at time $t = 6.80427E-05\,s$

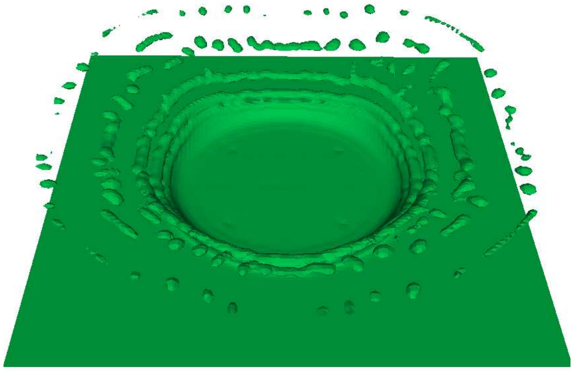

Figure 12: Droplet free surface at time $t = 8.78372E-05\,s$

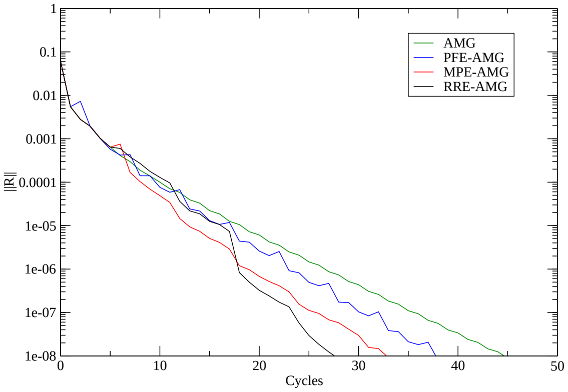

Figure 13: Free Surface AMG parameters: V-Cycle, Group Size 2, Gauss-Seidel smoother, 0 pre-sweeps and 2 post-sweeps.

Method	AMG Cycles	Time [s]
AMG	45	48.11
PFE-AMG	38	40.93
MPE-AMG	33	38.51
RRE-AMG	28	33.16

Table 7: Summary of results for Fig. (13).

methods. Unlike the residuals in the LES case, residual histories in this case are smoother allowing the extrapolation algorithms to perform well. Significant reductions in computational time as well as in the number of the AMG cycles required to achieve a reduction of the norm of the residual by 8 orders of magnitude are observed. More specifically, RRE-AMG is very efficient in bringing down both computational time and number of cycles regardless of what combinations of other controlling parameters are used.

The influence of the correction space dimension on MPE-AMG convergence properties is shown in figures

Method	AMG Cycles	Time [s]
AMG	47	63.39
PFE-AMG	37	48.21
MPE-AMG	41	58.90
RRE-AMG	24	36.13

Table 8: Summary of results for Fig. (14).

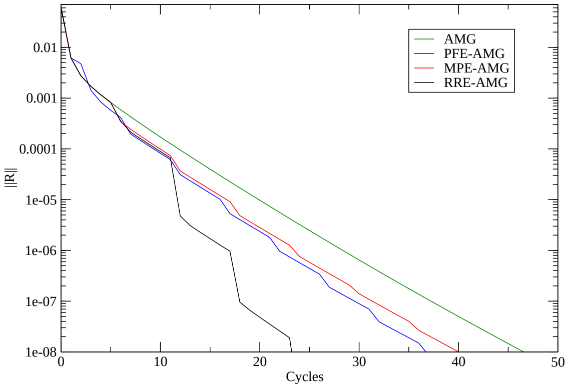

Figure 14: Free surface AMG parameters: W-Cycle, Group Size 4, ILU(0) smoother, 0 pre-sweeps and 2 post-sweeps.

Dimension	AMG Cycles	Time [s]
0	45	48.11
5	33	38.51
10	39	48.15
15	41	50.34

Table 9: Summary of results for Fig. (15).

(15) and (16). As before, increasing the size of the correction space does not bring substantial improvements The best results are obtained with the basis 5.

Similarly, the influence of the correction space dimension on RRE-AMG is shown in figures (17) and (18). Unlike the situation in MPE-AMG, increasing the size of the correction space basis has a significant influence on the number of iterations and the amount of computation time in RRE-AMG.

Dimension	AMG Cycles	Time [s]
0	47	63.39
5	41	58.90
10	41	59.95
15	43	63.60

Table 10: Summary of results for Fig. (16).

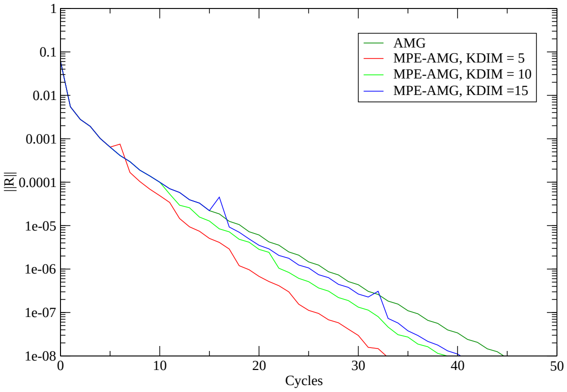

Figure 15: Free surface MPE-AMG parameters: V-Cycle, Group Size 2, Gauss-Seidel smoother, 0 pre-sweeps and 2 post-sweeps.

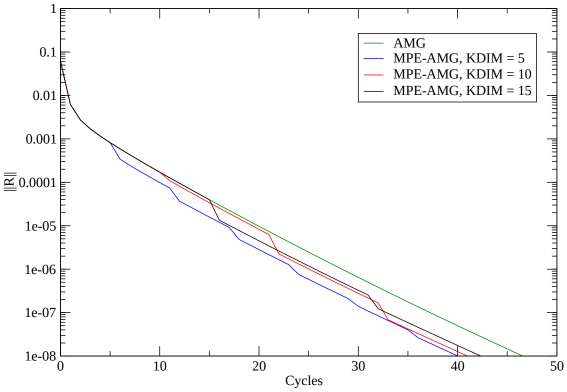

Figure 16: Free Surface MPE-AMG parameters: W-Cycle, Group Size 4, ILU(0) smoother, 0 pre-sweeps and 2 post-sweeps.

Dimension	AMG Cycles	Time [s]
0	45	48.11
5	33	38.51
10	31	37.46
15	36	46.08

Table 11: Summary of results for Fig. (17).

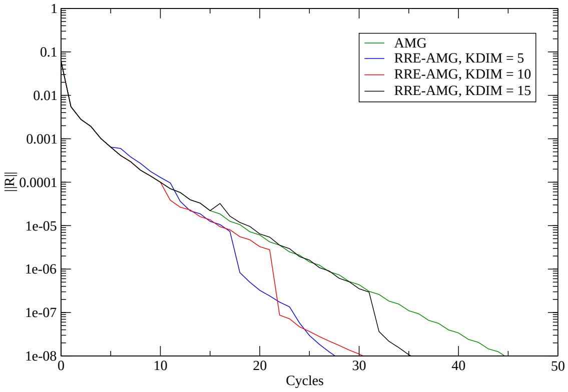

Figure 17: Free surface RRE-AMG parameters: V-Cycle, Group Size 2, Gauss-Seidel smoother, 0 pre-sweeps and 2 post-sweeps.

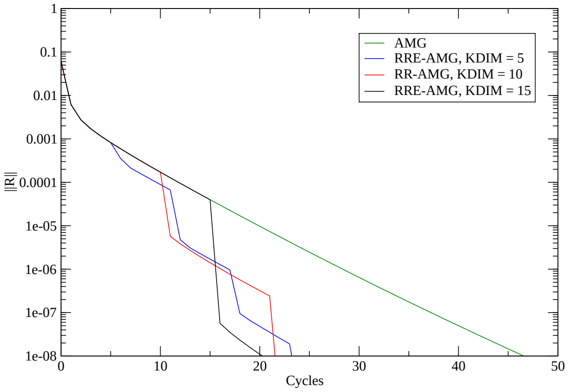

Figure 18: Free surface RRE-AMG parameters: W-Cycle, Group Size 4, ILU(0) smoother, 0 pre-sweeps and 2 post-sweeps.

Dimension	AMG Cycles	Time [s]
0	47	63.39
5	41	58.90
10	22	33.18
15	21	32.56

Table 12: Summary of results for Fig. (18).

8 CONCLUSION

Three new algorithms for performance improvements AMG are presented, namely PFE-AMG, MPE-AMG, and RRE-AMG. All three algorithms are based on extrapolation techniques. MPE-AMG and RRE-AMG correspond to preconditioned Krylov space methods, the Full Orthogonalization Method and the GMRES method respectively. Extrapolation in PFE-AMG is based on the ideas from singular perturbation theory and the disparity between eigenvalues in the matrix spectrum. New methods were applied to the solution linear systems of equations from two physically different time dependent problems, large eddy simulation over the forward facing step and tracking of free surface of the droplet during its impact on a flat wall. In both cases the new algorithms were applied to the solution of the pressure equation obtained from discretization of the incompressible Navier-Stokes equations by a cell-centered finite volume method. Parameters that influence the convergence behavior of all three methods are investigated including groups size in the AMG agglomeration, choice of the smoother, type of the AMG-Cycle and the size of the correction space (Krylov) dimension. Results show performance improvement ranging from 10% to 50% percent decrease in both the number of the AMG cycles and the computation time required to reduce the norm of the residual by 8 orders of magnitude. It is also shown that proper selection of the AMG parameters and correction space dimension result in a substantial increase in performance over the basic AMG method.

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