A Consistent Derivation of the Sea-Ice Model Using Conditional Averaging

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10th March 2003

Abstract

In this paper provides consistent derivation of a generic sea ice dynamics model using a mathematical technique called conditional averaging [Dopazo, 1977]. The model equations are derived directly from the mass and momentum balance for a continuum partially filled with ice. The new form of the model is then compared with the traditional form as given by Hibler [Hibler, 1979], with the objective of analysing the mathematical behaviour of the two models, specifically regarding the ice fraction equation. The conditional model implies a different and bounded form of transport, where the ice fraction equation is in a bounded non-conservative form. The derivation additionally provides hints on the form of wind forcing and ice-ocean interaction terms and the modelling of sub-grid scale ice interaction.

1 Introduction

A bulk of sea-ice modelling work today is based on the generic form of the sea ice transport model as described by Hibler [Hibler, 1979], with a number notable variants of rheological constitutive laws and ice distribution statistics. In particular, sea ice models with an elliptical viscous–plastic constitutive relation are found to reproduce observed ice drift well [Hutchings et al., 2002].

The work presented in this paper is a result of the author's frustration with the basic set of equations and their mathematical behaviour. In a recent work [Hutchings et al., 2002, Hutchings, 2000], the Hibler model has been implemented using the bounded and conservative Finite Volume (FV) method of discretisation, where some previously unseen inconsistencies have emerged. Moreover, in spite of my best efforts, I have been unable to track down the original derivation

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of the governing equations given in [Hibler, 1979]. Their exact form seems to have been accepted as standard by most practitioners in the field, leading to some mis-conceptions regarding the behaviour of the model and its, necessarily imperfect, numerical implementation. The basic premise of the work is that it is first necessary to examine the model in its differential form in detail (particularly in terms of boundedness and interaction between equations) before applying the necessary numerical modelling techniques.

In this paper, we shall examine the basic form of the Hibler model from a mathematical standpoint by deriving the governing equations from first principles using a technique called conditional averaging [Dopazo, 1977]. This is done in the expectation that the conditional model will yield the identical set of equations as the traditional form [Hibler, 1979].

Conditional averaging is a relatively novel mathematical tool used for the purpose of describing two- or multi-phase systems described as several intertwining continua. So far, it has been successfully implemented to various areas of continuum modelling, including multi-phase flows [Hill, 1998, Hill et al., 1994, Hill et al., 1995], combustion [Weller, 1993], astrophysics [Grulke et al., 2001] *etc.*

The rest of this paper is organised as follows. In Section 2 the viscous-plastic rheology model will be summarised and some general statements on its characteristics will be made. We shall then strip the model to its basic premises, excluding the thermodynamic source/sinks of ice and limiting ourselves to generic rheology. Section 3 describes the basic premise of the model derivation: a continuum partially filled with ice with leads of open water between ice floes and states the equations governing the behaviour of solid ice. In Section 3.1 we shall introduce the conditional averaging operators and state the transformations of various differential operators. The conditional averaging of ice equations will be performed in Section 4, finally stating the new conditional model. The model will be compared with the original model by Hibler and the mathematical characteristics of the two models will be examined in Section 5. The paper is completed with a short summary in Section 6.

2 Description of the Hibler Sea Ice Model

The viscous–plastic sea ice model was originally developed by Hibler [Hibler, 1979]. For daily time scales the momentum balance is taken to be:

$$\frac{\partial(m\mathbf{u})}{\partial t} + \nabla\bullet(m\mathbf{u}\mathbf{u}) = \mathbf{F} + \nabla\bullet\boldsymbol{\sigma}, \qquad (1)$$

where m is the ice mass per unit area, $\mathbf{u}$ is ice velocity and $\mathbf{F}$ is the sum of body and surface forces on the ice, including the wind stress and ocean stress on the ice surface, Coriolis force and force due to gravitational acceleration down the sea surface slope. The sub-grid scale ice interaction, $\boldsymbol{\sigma}$, is modelled as a viscous–plastic material obeying the associated flow rule and an elliptical yield criterion [Hibler, 1979].

$$\boldsymbol{\sigma} = \eta[\nabla\mathbf{u} + \nabla\mathbf{u}^T] + [(\zeta - \eta)\,\mathbf{I}\,\mathrm{tr}(\nabla\mathbf{u})] - \mathbf{I}\frac{P}{2}, \tag{2}$$

where the bulk (ζ) and shear (η) viscosities are given by

$$\zeta = \frac{P}{2\Delta}, \tag{3}$$

$$\eta = \frac{\zeta}{e^2} \tag{4}$$

and Δ is defined as

$$\Delta = \left[\mathrm{tr}\,(\dot{\boldsymbol{\varepsilon}})^2 + \frac{2}{e^2}\dot{\boldsymbol{\varepsilon}} : \dot{\boldsymbol{\varepsilon}}\right]^{\frac{1}{2}}. \tag{5}$$

Here, e is the ratio of semi-major to semi-minor axes of the elliptical yield curve, P is the ice strength, $\dot{\boldsymbol{\varepsilon}} = \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right]$ is the strain rate, $\mathbf{I}$ is the identity matrix and $\mathbf{q} : \mathbf{q}$ represents the scalar product of two second-rank tensors.

Ice mass continuity is described by the transport equations for effective ice thickness, $h = \frac{m}{\rho}$ where ρ is ice density, and A the ice fraction. Here, a two level thickness model [Hibler, 1979] is used and the sources of ice thickness, S_h, and area, S_A, are calculated from climatological growth/melt rates [Thorndike et al., 1975].

$$\frac{\partial h}{\partial t} + \nabla\!\bullet\!(h\mathbf{u}) = S_h, \tag{6}$$

$$\frac{\partial A}{\partial t} + \nabla\!\bullet\!(A\mathbf{u}) = S_A. \tag{7}$$

The thermodynamic source terms, S_h and S_A, are given as [Hibler, 1979]:

$$S_h = AF\left(\frac{h}{A}\right) + (1 - A)\,F\,(0)\,, \tag{8}$$

$$S_A = \min\left[\frac{A}{2h}S_h, 0\right] + \max\left[\frac{F(0)(1 - A)}{h_0}, 0\right]. \tag{9}$$

Here, F is a function describing the thermodynamic rate of change of ice thickness, taken from a look-up table [Thorndike et al., 1975] or modelled in some other way. Care should be taken to ensure S_h and S_A do not allow h or A to fall outside physical bounds [Hutchings, 2000]. Note that neither S_h nor S_A are a function $\mathbf{u}$.

To close the system of equations P must be determined. An empirical relationship for ice strength of the two level model is given by [Hibler, 1979]:

$$P = P^{\star} h\, e^{-C(1-A)}. \tag{10}$$

In further discussion, the exact form of the rheology and the thermodynamic coupling is not of interest, as it does not effect the derivation we are about to present. Without the loss of generality, we shall therefore limit the Hibler model to the following generic form:

$$\frac{\partial(m\mathbf{u})}{\partial t} + \nabla\bullet(m\mathbf{u}\mathbf{u}) = \nabla\bullet\boldsymbol{\sigma}, \tag{11}$$

$$\frac{\partial h}{\partial t} + \nabla\bullet(h\mathbf{u}) = 0, \tag{12}$$

$$\frac{\partial A}{\partial t} + \nabla\bullet(A\mathbf{u}) = 0. \tag{13}$$

The necessary sources/sinks and rheology model may be added later.

3 Derivation of the Sea Ice Transport Model

In this section, we shall attempt to derive the equivalent of Eqs. (11, 12 and 13) from first principles. For this purpose, we need to postulate that the behaviour of sea ice will be examined on the scale larger than the scale of individual ice floes and the leads between them. In this way, the lowest-level control volume of interest is assumed to contain both the ice and the free water and the behaviour of the system is averaged over it. The limits of "ice-only" and "open water" are naturally included in the derivation.

We can state the governing equations for the ice phase:

- Conservation of mass ($\rho = const.$):

  $$\frac{\partial \rho}{\partial t} + \nabla\bullet(\rho\mathbf{u}) = 0, \tag{14}$$

  where ρ is the density of ice;

- Conservation of momentum:

  $$\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla\bullet(\rho\mathbf{u}\mathbf{u}) = \nabla\bullet\boldsymbol{\sigma}. \tag{15}$$

Note that $\mathbf{u}$ and $\boldsymbol{\sigma}$ are equal to the ice velocity and ice interaction tensor where the ice is present and undefined in the absence of ice.

3.1 Conditional Averaging

Conditional averaging is applied to the sea ice system, consisting of ice floes and open water by considering separately the ice, the open water and the interface between them. Consider a control volume containing a sea ice fragment and a lead of open water, Fig. 1.

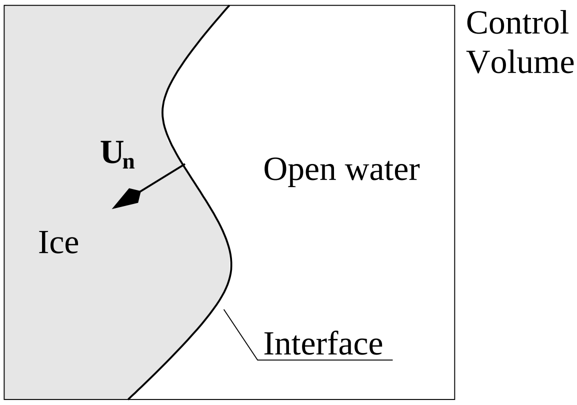

Figure 1: Control volume containing ice and open water.

The velocity of the ice-water interface is denoted $\mathbf{u}_n$ and we shall denote the ice velocity with $\mathbf{u}$, keeping them independent at the moment.

Additionally, we will need to describe the distribution of the ice floes in the domain, including the sub-grid scale. For this purpose, we shall introduce the indicator function $I(\mathbf{x}, t)$ as

$$I(\mathbf{x}, t) = \begin{cases} 1 \text{ if point } (\mathbf{x}, t) \text{ is covered by ice,} \\ 0 \text{ if point } (\mathbf{x}, t) \text{ is in open water.} \end{cases} \tag{16}$$

Equations are conditionally averaged [Weller, 1993] by multiplying by the indicator function I and then applying conventional averaging techniques. This approach is a simple extension of that applied to intermittent turbulent flows by Dopazo [Dopazo, 1977]. With the use of the indicator function, and by allowing the phase interface to propagate, the analytical methods developed by Dopazo can be applied to the present problem [Dopazo, 1977].

The indicator function can be viewed in several ways, the most intuitive of which is that its average over the control volume represents the area fraction of ice per unit area, *i.e.* $\gamma = A$ from Eqn. (13):

$$\gamma = \overline{I(\mathbf{x}, t)}, \tag{17}$$

where the over-bar denotes the ensemble average. γ can also be seen as the probability of point $(\mathbf{x}, t)$ being covered by ice.

3.2 Conditional Averaging Operators

Let $\mathbf{Q}(\mathbf{x}, t)$ be any physical property, scalar or tensor of any rank. It then follows:

$$\overline{\nabla \mathbf{Q} I} = \nabla(\overline{\mathbf{Q} I}) = \nabla(\gamma \overline{\mathbf{Q}}_\gamma), \tag{18}$$

where $\overline{\mathbf{Q}}_\gamma$ denotes the conditional average $\mathbf{Q}$ in zone 1, denoted by subscript γ. Also:

$$\overline{\frac{\partial \mathbf{Q}I}{\partial t}} = \frac{\partial(\overline{\mathbf{Q}I})}{\partial t} = \frac{\partial(\gamma\overline{\mathbf{Q}}_\gamma)}{\partial t}. \tag{19}$$

Consider the mean of $I\nabla\mathbf{Q} = \nabla(\mathbf{Q}I) - \mathbf{Q}\nabla I$ over the control volume δV. Note ∇I is non-zero only at the interface, where it has the absolute value of the Dirac delta function and the direction of the unit normal $\mathbf{n}_\gamma$ to the interface, pointing into zone 1. Then:

$$\overline{I\nabla\mathbf{Q}} = \nabla(\overline{\mathbf{Q}I}) - \lim_{\delta V\to 0}\overline{\frac{1}{\delta V}\int_{S(\mathbf{x},\,t)}\mathbf{Q}\mathbf{n}_\gamma(\mathbf{x},\,t)\,dS}, \tag{20}$$

where $S(\mathbf{x}, t) = 0$ is the equation for the interface. We define the surface average $\widehat{\mathbf{Q}}$ of a property $\mathbf{Q}$ as the surface integral per unit volume divided by the surface area per unit volume Σ:

$$\widehat{\mathbf{Q}} \equiv \frac{\lim_{\delta V\to 0}\overline{\frac{1}{\delta V}\int_{S(\mathbf{x},\,t)}\mathbf{Q}(\mathbf{x},\,t)\,dS}}{\Sigma}, \tag{21}$$

where

$$\Sigma = \lim_{\delta V\to 0}\overline{\frac{1}{\delta V}\int_{\mathbf{S}(\mathbf{x},\,t)}d\mathbf{S}}. \tag{22}$$

Eqn. (20) may then be re-written as:

$$\overline{I\nabla\mathbf{Q}} = \nabla(\gamma\overline{\mathbf{Q}}_\gamma) - \widehat{\mathbf{Q}\mathbf{n}_\gamma}\Sigma. \tag{23}$$

Also:

$$\overline{I\nabla\bullet\mathbf{Q}} = \nabla\bullet(\gamma\overline{\mathbf{Q}}_\gamma) - \widehat{\mathbf{Q}\bullet\mathbf{n}_\gamma}\Sigma \tag{24}$$

and

$$\overline{I\frac{\partial\mathbf{Q}}{\partial t}} = \frac{\partial(\gamma\overline{\mathbf{Q}}_\gamma)}{\partial t} + \widehat{\mathbf{Q}\mathbf{u}_s\bullet\mathbf{n}_\gamma}\Sigma, \tag{25}$$

where $\mathbf{u}_s$ is the velocity of the interface.

4 Conditional Sea Ice Model

We shall now apply the conditional averaging techniques to the sea ice model by choosing $I = 1$ for the area covered by ice and $I = 0$ for open water. The derivation is made easier by the fact that the "properties" open water (in the sea ice modelling sense) will be neglected, *i.e.* we are dealing with a single material which does not completely fill the domain of interest. In addition, an equation representing the average position of the interface (or ice distribution) is also required.

4.1 Ice Fraction Equation

Substituting $\mathbf{Q} = 1$ into Eqn. (25) yields:

$$\frac{\partial \gamma}{\partial t} = -\widetilde{\overline{\mathbf{u}_s \bullet \mathbf{n}_\gamma}} \Sigma. \tag{26}$$

Decomposing $\widetilde{\overline{\mathbf{u}_s \bullet \mathbf{n}_\gamma}}$ into surface averages and surface fluctuation correlations strictly leads to:

$$\widetilde{\overline{\mathbf{u}_s \bullet \mathbf{n}_\gamma}} = \widetilde{\overline{\mathbf{u}_s}} \bullet \widetilde{\overline{\mathbf{n}_\gamma}} + \widetilde{\overline{\mathbf{u}_s{}^\sharp \bullet \mathbf{n}_\gamma{}^\sharp}} \tag{27}$$

In our case, the (ensemble) surface fluctuations are of no interest and will therefore be neglected: $\widetilde{\overline{\mathbf{u}_s{}^\sharp \bullet \mathbf{n}_\gamma{}^\sharp}} = 0$. It remains to model the velocity of the interface between the ice and open water averaged over the surface. Even under extreme conditions it is reasonable to say that the motion of the ice floe is considerably faster than the interface shift caused by the freezing/melting of ice in contact with water: we shall therefore assume that the ice edge moves with the same velocity $\mathbf{u}$ as the ice itself, both instantaneously and in its average value[1]:

$$\mathbf{u}_s = \mathbf{u}, \tag{28}$$

$$\widetilde{\overline{\mathbf{u}_s}} = \overline{\mathbf{u}_\gamma}. \tag{29}$$

A similar model has successfully been applied by Weller [Weller, 1993] in modelling of turbulent flames. In our case, the situation is much simpler because the interface is not fluctuating in nature.

A further manipulation of Eqn. (26) can be introduced by substituting $\mathbf{Q} = 1$ into Eqn. (23):

$$\widetilde{\overline{\mathbf{n}_\gamma}} \Sigma = \nabla \gamma, \tag{30}$$

yielding the following form of the interface equation:

$$\frac{\partial \gamma}{\partial t} + \overline{\mathbf{u}_\gamma} \bullet \nabla \gamma = 0. \tag{31}$$

4.2 Conditional Ice Mass Equation

The mass conservation equation for the ice, Eqn. (14), reads:

$$\frac{\partial \rho}{\partial t} + \nabla \bullet (\rho \mathbf{u}) = 0.$$

Multiplying Eqn. (14) with the indicator function and performing the averaging yields:

$$\overline{I \frac{\partial \rho}{\partial t}} + \overline{I \nabla \bullet (\rho \mathbf{u})} = 0. \tag{32}$$

[1]Note that we are replacing the surface average with a volume average around the surface. This, however, proves to be a good model as the volume over which the averaging is performed can be shrunk onto the surface since the surface is effectively free of fluctuations.

The above terms can be manipulated as follows:

$$\overline{I\frac{\partial \rho}{\partial t}} = \frac{\partial(\gamma\overline{\rho_\gamma})}{\partial t} + \widetilde{\rho \mathbf{u}_s \bullet \mathbf{n}_\gamma}\Sigma, \tag{33}$$

$$\overline{I\nabla\bullet(\rho\mathbf{u})} = \nabla\bullet(\gamma\overline{\rho_\gamma}\,\overline{\mathbf{u}_\gamma}) - \widetilde{\rho\mathbf{u}\bullet\mathbf{n}_\gamma}\Sigma. \tag{34}$$

Applying the surface velocity model, Eqn. (28), the surface-averaged terms cancel out, yielding:

$$\frac{\partial(\gamma\overline{\rho_\gamma})}{\partial t} + \nabla\bullet(\gamma\overline{\rho_\gamma}\,\overline{\mathbf{u}_\gamma}) = 0. \tag{35}$$

4.3 Ice Momentum Equation

The derivation of the ice momentum equation starts from Eqn. (15):

$$\frac{\partial(\rho\mathbf{u})}{\partial t} + \nabla\bullet(\rho\mathbf{u}\mathbf{u}) = \nabla\bullet\boldsymbol{\sigma}.$$

Multiplying the above with the indicator function and performing the averaging, we obtain:

$$\overline{I\frac{\partial(\rho\mathbf{u})}{\partial t}} + \overline{I\nabla\bullet(\rho\mathbf{u}\mathbf{u})} = \overline{I\nabla\bullet\boldsymbol{\sigma}}. \tag{36}$$

The following term-by-term transformations may now be performed:

$$\overline{I\frac{\partial(\rho\mathbf{u})}{\partial t}} = \frac{\partial(\gamma\overline{\rho_\gamma}\,\overline{\mathbf{u}_\gamma})}{\partial t} + \widetilde{\rho\mathbf{u}\mathbf{u}_s\bullet\mathbf{n}_\gamma}\Sigma, \tag{37}$$

$$\overline{I\nabla\bullet(\rho\mathbf{u}\mathbf{u})} = \nabla\bullet(\gamma\overline{\rho_\gamma}\,\overline{\mathbf{u}_\gamma}\,\overline{\mathbf{u}_\gamma}) - \widetilde{\rho\mathbf{u}\mathbf{u}\bullet\mathbf{n}_\gamma}\Sigma, \tag{38}$$

$$\overline{I\nabla\bullet\boldsymbol{\sigma}} = \nabla\bullet(\gamma\overline{\boldsymbol{\sigma}}_\gamma) - \widetilde{\boldsymbol{\sigma}\bullet\mathbf{n}_\gamma}\Sigma. \tag{39}$$

Note that the last term in Eqn. (39) represents the surface-averaged normal stress on the edge of the ice floe, *i.e.* the interaction between the ice and water. This is typically incorporated into the ocean forcing and we shall therefore consider it a part of a given external forcing term[2]. The actual form of this term should, however, be taken into account when modelling the ice-ocean interaction.

Assembling the above and using the interfacial velocity model, Eqn. (28), the final form of the conditional momentum equation reads:

$$\frac{\partial(\gamma\overline{\rho_\gamma}\,\overline{\mathbf{u}_\gamma})}{\partial t} + \nabla\bullet(\gamma\overline{\rho_\gamma}\,\overline{\mathbf{u}_\gamma}\,\overline{\mathbf{u}_\gamma}) = \nabla\bullet(\gamma\overline{\boldsymbol{\sigma}}_\gamma). \tag{40}$$

[2]Note that this terms represents the ice-ocean interaction in the 2-D sense, *i.e.* at the edges of the ice floe. The drag on the floe also appears along its surface and in not present in this term.

4.4 Effect of Interfacial Modelling

During this derivation, we had to introduce some modelling, which can, in principle, be done in a number of ways. It is therefore necessary to examine the effect of the choices we made to the final form of the model. As a guidance, the modelling has been chosen to drive the conditional model towards the well-established traditional form and to be as intuitive as possible.

The modelled terms first appear in Eqn. (27) and then in a similar form in Eqs. (33, 34, 37 and 38). However, the modelling for the terms pairs of equation in Eqs. (33 and 34) and Eqs. (37 and 38) is such that the pairs of surface terms cancel out, which is intuitively correct.

4.4.1 Modelling Interfacial Velocity

In the interfacial velocity model, we state that the edge of an ice floe moves with the same velocity as the ice floe itself. Several other choices (having in mind the averaging that is being performed and the possible presence of sub-scale oscillations) may be done, but this is the simplest and most intuitive. Moreover, as will be shown later, the crucial result of the derivation is the non-conservative form of the γ equation and this is independent of the modelling: combining Eqn. (26) and Eqn. (30) (which follows directly from Eqn. (23)), we obtain:

$$\frac{\partial \gamma}{\partial t} + \widehat{\mathbf{u}_s} \bullet \nabla \gamma + \overbrace{\mathbf{u}_s{}^{\sharp} \bullet \mathbf{n}_\gamma{}^{\sharp}} \Sigma = 0. \tag{41}$$

From the above it is clear that the convection of γ will be non-conservative irrespective of the choice of model for $\widehat{\mathbf{u}_s}$.

4.4.2 Modelling Surface Fluctuations

In the derivation of the conditional model we have chosen to neglect the sub-scale surface fluctuation of the interface are considered to be negligible. Terms of this type are typically modelled as "turbulent diffusion" [Weller, 1993] and such terms are also absent from the traditional model.

4.5 Final Form of the Conditional Model

All the components of the conditional model are now available. Before we state the model in its final form, we can somewhat simplify the notation. As noted before, $\mathbf{Q}_\gamma$ represents the property $\mathbf{Q}$ in the zone denoted by γ. In our case, we are dealing with only a single component (ice) and the open water is treated as cavity. Thus, the average ice mass per unit area, in the absence of the "second phase" and noting that $\rho = const.$ is defined as:

$$m = \gamma \overline{\rho_\gamma} + (1 - \gamma)0 = \gamma \rho \tag{42}$$

Collecting Eqs. (31, 35 and 40) and substituting the above where necessary leads to the final form of the conditional model:

$$\frac{\partial \gamma}{\partial t} + \overline{\mathbf{u}_\gamma} \bullet \nabla \gamma = 0,$$

$$\frac{\partial m}{\partial t} + \nabla \bullet (m \overline{\mathbf{u}_\gamma}) = 0, \tag{43}$$

$$\frac{\partial (m \overline{\mathbf{u}_\gamma})}{\partial t} + \nabla \bullet (m \overline{\mathbf{u}_\gamma}\, \overline{\mathbf{u}_\gamma}) = \nabla \bullet (\gamma \overline{\boldsymbol{\sigma}}_\gamma) \tag{44}$$

5 Model Comparison

Let us now compare the conditional model with the original model by Hibler, Eqs. (11, 12 and 13), looking for similarities.

- Eqn. (31) is the interfacial equation and represents the distribution of ice area per unit area (or ensemble probability of ice presence) in the domain. Thus, its function is equivalent to Eqn. (13) in the Hibler model but its form of convection transport is considerably different.
- Eqn. (12) is derived from the conditional mass conservation and is in every detail identical to Eqn. (12) of the Hibler model.
- Eqn. (44) represents the momentum balance and in its form is equivalent to Eqn. (11) of the Hibler model.

The main difference between the two models is the form of the interfacial equation, which has been the initial concern behind this study. Firstly, it is clear from the derivation that γ in Eqn. (31) is not a conserved property (it is derived from the indicator function). The resulting equation is therefore in a non-conservative form. Secondly, it is clear that γ always remains bounded between zero and unity irrespective of the velocity field and the equation should therefore remain sourceless.

In the Hibler model, the A equation reads:

$$\frac{\partial A}{\partial t} + \nabla \bullet (A \mathbf{u}) = 0,$$

compared to the form from the conditional model ($\gamma \equiv A$ and $\overline{\mathbf{u}_\gamma} \equiv \mathbf{u}$):

$$\frac{\partial \gamma}{\partial t} + \overline{\mathbf{u}_\gamma} \bullet \nabla \gamma = 0.$$

In terms of boundedness, both γ and A are bounded below by zero but the A-equation may produce the values of $A > 1$ in convergent flow, which is physically impossible. If both model derivations are correct and start from identical premises, the final form of the model (barring the differences in modelling) should be identical. Therefore, we propose that the Hibler model should be modified by adding an additional source term equal to $A\nabla\bullet\mathbf{u}$ to ensure A is bounded above by unity.

Finally, a few words on the numerical implementation. The original paper describing the Hibler model also includes the details of a doubly staggered finite difference scheme used to discretise it. This kind of numerics is inherently unbounded (*i.e.* the discretised form of the equations does not preserved the natural bounds of the differential form) and it was necessary to introduce cut-offs on both h and A to achieve the solution. The issue with the exact form of the A equation has come to light during the implementation of the Finite Volume (FV) sea-ice model [Hutchings et al., 2002] where it was necessary to change the form of transport in the A equation to make the model behave well. We hope the derivation of the conditional form of the model, as described above, represents a sufficient argument for this change.

6 Summary

This paper presented a new and consistent derivation of the sea-ice model from first principles using conditional averaging. The final form of the model is very similar to the model proposed by Hibler [Hibler, 1979], with a notable difference in the form of the ice distribution equation. The interface equation in the new model possesses the correct mathematical behaviour in terms of boundedness, on the basis of which a modification in the form of transport for A has been proposed. The incorporation of the thermodynamical and interaction sources/sinks in the conditional averaged model is left as an exercise to the reader – the expectation is that that wind and ocean drag terms also depend on the ice fraction, which is absent in the traditional model.

Acknowledgements

I would like to thank dr. Jenny Hutchings for her continuous doubts about the form of transport in the sea ice model. Help on conditional averaging from Henry Weller is also gratefully acknowledged.

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